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THE GROUP $G_1(RG)$ FOR A NILPOTENT GROUP G

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0. Introduction

Let R be a right noetherian domain and K be its quotient field with $\mathrm{ch}(K)=0$. Recently, for a finite abelian group G, M. Liu [3] got a formula of $G_1(RG)$ (of [1, p. 453], which does not coincide with Quillen's $K_1(\mathcal{M}_{RG})$), which is a generalization of the beautiful H. Lenstra's calculation of $G_0(RG)$, (see [2]). Namely, he got the following isomorphism:

$$G_1(RG)\cong \bigoplus_{\varrho\in X(G)} G_1(R(\varrho))/H_\varrho \tag{0.1}$$

where $X(G)$ is the set of cyclic quotient groups of G and H_ϱ will be described below. On the other hand, the author [4] generalized the formula of H. Lenstra into a nilpotent group. Using similar arguments, we will show that a similar formula as M. Lui's holds for a nilpotent group.

Now let G be a finite nilpotent group and write $G=\prod_p G_p$ as the direct product of its Sylow p-subgroups G_p. Let Y be the set of representatives for the K-conjugacy classes of irreducible characters of G and $e(\theta)$ denote the central primitive idempotent of KG corresponding to $\theta\in Y$. Furthermore, let $\pi(G)$ and $\pi(\theta)$ be the sets of all prime divisors of the order of G and the order $n(\theta)$ of $G/\mathrm{Ker}\,\theta$, respectively. Since $\mathcal{M}_{RGe(\theta)/n(\theta)RGe(\theta)}$ can be considered to be a subcategory of $\mathcal{M}_{RGe(\theta)}$, there is a homomorphism:

$$G_1(RGe(\theta)/n(\theta)RGe(\theta))\to G_1(RGe(\theta)).$$

Let H_θ be the image of this homomorphism. Similarly, H_ϱ in (0.1) was defined in [3]. Thus $G_1(RGe(\theta))/H_\theta$ is presented by adding to the definition of $G_1(RGe(\theta))$ the additional relations $[M,\alpha]=0$ whenever $n(\theta)M=0$.

Theorem. $G_1(RG)\cong\bigoplus_{\theta\in Y} G_1(RGe(\theta))/H_\theta$.

For a set S of primes, set $G_S=\prod_{p\in S}G_p$ and θ_S denotes an irreducible constituent of θ_{G_S}. Since θ_{G_S} is homogeneous, θ_S is well-defined. The canonical homomorphisms $G \to G_S \to G$ induce, by restriction, an exact functor N_S (see [2]).

1. The homomorphism $\Phi : \bigoplus_{\theta \in Y} G_1(RGe(\theta))/H_\theta \to G_1(RG)$

For $\theta \in Y$, $M \in \mathcal{M}_{RGe(\theta)}$, and $\alpha \in \mathrm{Aut}_{RGe(\theta)}(M)$, we will write

$$[M, \alpha; \theta] = \text{class of } (M, \alpha) \text{ in } G_1(RGe(\theta)),$$

$$[M, \alpha; \langle\theta\rangle] = \text{class of } (M, \alpha) \text{ in } G_1(RGe(\theta))/H_\theta,$$

$$[M, \alpha; G] = \text{class of } (M, \alpha) \text{ in } G_1(RG)$$

where we embed $\mathcal{M}_{RGe(\theta)}$ in $\mathcal{M}_{RG}$ via the canonical projection

$$RG \to RG/(RG \cap KG(1 - e(\theta))) \cong RGe(\theta).$$

We define $\phi'_\theta : G_1(RGe(\theta)) \to G_1(RG)$ by

$$\phi'_\theta[M, \alpha; \theta] = \sum_{S \subseteq \pi(\theta)} (-1)^{\#(\pi(\theta) - S)}[N_S M, \alpha; G].$$

Since every RG-module N with $pN = 0$ has a nonzero $\mathrm{End}_{RG}(N)$-invariant RG-submodule on which G_p acts trivially, by applying the same argument in [3], we have that $\phi'_\theta(H_\theta) = 0$. Namely, we can get a homomorphism

$$\Phi : \bigoplus_{\theta \in Y} G_1(RGe(\theta))/H_\theta \to G_1(RG).$$

2. The inverse $\Psi : G_1(RG) \to \bigoplus_{\theta \in Y} G_1(RGe(\theta))/H_\theta$

Let $\theta \in Y$ and let S be a set of primes. The functor N_S carries the subcategory $\mathcal{M}_{RGe(\theta)}$ to $\mathcal{M}_{RGe(\theta_S)}$. Thus we can define

$$\psi_\theta : G_1(RGe(\theta)) \to \bigoplus_{\chi \in Y} G_1(RGe(\chi))/H_\chi$$

by $\psi_\theta[M, \alpha; \theta] = \sum_{S \subseteq \pi(\theta)} [N_S M, \alpha; \langle\theta\rangle]$.

Lemma 1. *Let $M \in \mathcal{M}_{RG}$. Then there is a chain of submodules $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0$ such that, for each i, M_i is stable under* $\mathrm{End}_{RG}(M)$, *and $M_i/M_{i+1} \in \mathcal{M}_{RGe(\theta)}$ for some $\theta \in Y$.*

Except for the assertion about $\mathrm{End}_{RG}(M)$-invariant, this is just Lemma 2 in [4]. Moreover, since $M_i = \prod_{j=1}^{i} (1 - e(\theta_j))M$ in [4] is invariant under $\mathrm{End}_{RG}(M)$.

Let $M \in \mathcal{M}_{RG}$ and $\alpha \in \mathrm{Aut}_{RG}(M)$. With the notation of Lemma 1, let $\alpha_i \in \mathrm{Aut}_{RG}(M_i/M_{i+1})$ be the automorphism induced by α, and choose $\theta_i \in Y$ so that $M_i/M_{i+1} \in \mathcal{M}_{RGe(\theta_i)}$. Put

$$\Psi[M, \alpha; G] = \sum_{i=0}^{t-1} \psi_{\theta_i}[M_i/M_{i+1}, \alpha_i; \langle\theta_i\rangle].$$

By Lemmas 1 and 3 in [4], this is independent of the choice of the θ_i's. Moreover, Ψ does not depend on the filtration of M, that is, Ψ is well-defined. To see that Φ and Ψ are the inverses of each other, we suffice to follow the calculation in [2].

This concludes the proof of the Theorem.

References

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